

UNIVERSAL STRING AND SMALL $N = 4$ SUPERSTRINGNobuyoshi Ohta¹ and Takashi Shimizu²*Department of Physics, Osaka University, Toyonaka, Osaka 560, Japan***Abstract**

It was previously shown that most of the superstrings can be obtained from those with higher world-sheet supersymmetry as spontaneously broken phases. In this paper, we show that the small $N = 4$ superstring, which was left out of this hierarchy of the universal string, can be obtained from the large $N = 4$ strings. We also show that the $N = 2$ string is a special vacuum of small $N = 4$ string. Thus all the known superstring theories can be derived from a universal string by spontaneous breakdown of supersymmetry.

¹e-mail address: ohta@phys.wani.osaka-u.ac.jp

²e-mail address: simtak@phys.wani.osaka-u.ac.jp

Recently it has been shown that any $N = 0$ ($N = 1$) string can be obtained from $N = 1$ ($N = 2$) superstring as a spontaneously broken phase [1]. This remarkable discovery has opened the possibility of constructing a universal string from which all known string theories can be derived just by selecting different vacua. Indeed this has been further generalized, and it is now known that an arbitrary N -extended superstring can be regarded as a special class of vacua for $(N + 1)$ superstring [2].

At the moment, this hierarchy has been successfully formulated for N -extended superstrings based on linear superconformal algebras (SCAs) with $SO(N)$ current algebra [3]. However there is an important class of superstrings which is usually termed the ‘small’ $N = 4$ superstring with only $SU(2)$ current algebra. This class of superstrings is the largest extended one that has been formulated by lagrangian and also constitutes an interesting theory from the point of view of topological string. It is thus important to examine if this class of superstrings also belongs to the hierarchy of the universal string or if it is completely decoupled from such a unified approach.

The purpose of this paper is to show that it is in fact possible to embed the small $N = 4$ superstring into the hierarchy. We will show that the $N = 2$ superstring is realized as a special vacuum of the small $N = 4$ superstring, which in turn is realized in the large $N = 4$ superstring. We also point out that the large $N = 4$ superstring with an arbitrary parameter x (the ratio of the levels of $SU(2)$ current algebras contained in the theory) is realized in the already existing embeddings. This completes the program of the universal string concerning the linear SCA [2].

In order to discuss the embeddings of some superstring theory into the small $N = 4$, we must first ask what theory can be embedded into the small $N = 4$ string. This can be easily answered if one notices that the $N = 3$ superstring contains a symmetry generator of dimension $\frac{1}{2}$ which is not present in the small $N = 4$ superstring. Consequently the largest super-subalgebra of the small $N = 4$ SCA is the $N = 2$ SCA. We thus conclude that it is only possible to embed the $N = 2$ superstring into the small $N = 4$ string.

Let us first discuss the embedding of $N = 2$ superstrings into the small $N = 4$ one. For this purpose, we use $N = 2$ superfields. Thus the small $N = 4$ SCA can be written

in the $N = 2$ superfields as follows:

$$\begin{aligned}
T(Z_1)T(Z_2) &\sim \frac{\frac{1}{3}c + \theta_{12}\bar{\theta}_{12}T}{z_{12}^2} + \frac{-\theta_{12}DT + \bar{\theta}_{12}\bar{D}T + \theta_{12}\bar{\theta}_{12}\partial T}{z_{12}}, \\
T(Z_1)G_c(Z_2) &\sim \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} + \frac{2}{z_{12}} \right) G_c + \frac{\bar{\theta}_{12}\bar{D}G_c + \theta_{12}\bar{\theta}_{12}\partial G_c}{z_{12}}, \\
T(Z_1)G_a(Z_2) &\sim \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} - \frac{2}{z_{12}} \right) G_a + \frac{-\theta_{12}DG_a + \theta_{12}\bar{\theta}_{12}\partial G_a}{z_{12}}, \\
G_c(Z_1)G_a(Z_2) &\sim -\frac{c}{6} \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^3} - \frac{1}{z_{12}^2} \right) - \frac{1}{2} \left(\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} - \frac{2}{z_{12}} \right) T + \frac{\bar{\theta}_{12}}{z_{12}} \bar{D}T, \quad (1)
\end{aligned}$$

where G_c and G_a are chiral and antichiral superfields, respectively. Here $Z \equiv (z, \theta, \bar{\theta})$, and $D \equiv \partial_\theta - \bar{\theta}_2^1 \partial_z$, $\bar{D} \equiv \partial_{\bar{\theta}} - \theta_2^1 \partial_z$ are the covariant derivatives, and $z_{12} \equiv z_1 - z_2 + \frac{1}{2}(\theta_1 \bar{\theta}_2 + \bar{\theta}_1 \theta_2)$, $\theta_{12} \equiv \theta_1 - \bar{\theta}_2$, $\bar{\theta}_{12} \equiv \bar{\theta}_1 - \theta_2$. The $N = 2$ subalgebra is given by the first line of eq. (1) with the generator T alone.

In order to embed the $N = 2$ superstring into the small $N = 4$, we take an arbitrary background T_m for the $N = 2$ superstring which satisfies the first relation in eq. (1). We then introduce two sets of chiral and antichiral fields (ξ_c, β_c) , (ξ_a, β_a) , corresponding to the generators G_c and G_a . Here ξ_c, ξ_a are anticommuting fields of spin $-\frac{1}{2}$ while β_c, β_a are commuting ones of spin 1 with the operator product expansion (OPE)

$$\xi_c(Z_1)\beta_c(Z_2) \sim \frac{\bar{\theta}_{12}}{z_{12}}, \quad \xi_a(Z_1)\beta_a(Z_2) \sim \frac{\theta_{12}}{z_{12}}. \quad (2)$$

These fields are regarded as part of matter fields of our theory. We then find the following (non-critical) realization satisfies the small $N = 4$ SCA (1) with the central charge shifted to $c - 18$:

$$\begin{aligned}
T &= T_m + 2\bar{D}\xi_c\beta_c - \xi_c\bar{D}\beta_c - 2D\xi_a\beta_a + \xi_a\beta_a, \\
G_c &= \beta_c + D \left[\xi_a T_m - \xi_a \xi_c \bar{D}\beta_c + 2\xi_a \bar{D}\xi_c \beta_c - \xi_a D\xi_a \beta_a + \frac{18-c}{6} \partial \xi_a \right], \\
G_a &= \beta_a. \quad (3)
\end{aligned}$$

The critical algebra is for $c = 6$. However the non-critical realization may be useful in analyzing the properties of topological strings.

It should be noted that the structure of the additional super-generator G_c is completely different from the known embeddings [1, 2]. In these cases, it was obtained just

by “improving” the BRST current by total derivatives such that it satisfies the correct OPE. However our generator G_c has a fundamentally different structure from the BRST current.

Substituting the generators (3) for $c = 6$ into the BRST operator for the small $N = 4$ string

$$Q_{sN4} = \oint \frac{dzd^2\theta}{2\pi i} \left[C_t \left(T + \frac{1}{2} \partial C_t B_t + \frac{1}{2} DC_t \bar{D} B_t + \frac{1}{2} \bar{D} C_t D B_t + 2DC_a B_a + C_a D B_a - 2\bar{D} C_c B_c - C_c \bar{D} B_c \right) - B_t C_c C_a \right] + \oint \frac{dzd\bar{\theta}}{2\pi i} C_c G_c + \oint \frac{dzd\theta}{2\pi i} C_a G_a, \quad (4)$$

we find

$$Q_{sN4} = \oint \frac{dzd^2\theta}{2\pi i} \left[C_t \left(T_m + \frac{1}{2} \partial C_t B_t + \frac{1}{2} DC_t \bar{D} B_t + \frac{1}{2} \bar{D} C_t D B_t + 2DC_a B_a + C_a D B_a - 2\bar{D} C_c B_c - C_c \bar{D} B_c - 2D\xi_a \beta_a + \xi_a D\beta_a + 2\bar{D}\xi_c \beta_c - \xi_c \bar{D}\beta_c \right) - B_t C_c C_a + C_c \xi_a T_m - C_c \xi_a \xi_c \bar{D}\beta_c + 2C_c \xi_a \bar{D}\xi_c \beta_c - C_c \xi_a D\xi_a \beta_a + 2C_c \partial \xi_a \right] + \oint \frac{dzd\bar{\theta}}{2\pi i} C_c \beta_c + \oint \frac{dzd\theta}{2\pi i} C_a \beta_a. \quad (5)$$

We can show the equivalence of this particular class of the small $N = 4$ string to the $N = 2$ string. Our method to show this is to use a similarity transformation that maps this BRST operator (5) for the small $N = 4$ superstring to a sum of those for the $N = 2$ and topological sectors [4]. We find that the following similarity transformation does this job:

$$e^{R_3} e^{R_2} e^{R_1} Q_{sN4} e^{-R_1} e^{-R_2} e^{-R_3} = Q_{N=2} + Q_{top}, \quad (6)$$

where

$$Q_{N=2} = \oint \frac{dzd^2\theta}{2\pi i} C_t \left(T_m + \frac{1}{2} \partial C_t B_t + \frac{1}{2} DC_t \bar{D} B_t + \frac{1}{2} \bar{D} C_t D B_t \right), \\ Q_{top} = \oint \frac{dzd\bar{\theta}}{2\pi i} C_c \beta_c + \oint \frac{dzd\theta}{2\pi i} C_a \beta_a, \quad (7)$$

and where

$$R_1 = - \oint \frac{dzd^2\theta}{2\pi i} C_c \xi_a B_t, \\ R_2 = - \oint \frac{dzd^2\theta}{2\pi i} C_c \xi_a D\xi_a B_a, \\ R_3 = \oint \frac{dzd^2\theta}{2\pi i} [DC_t \xi_a B_a + C_t D\xi_a B_a - \bar{D} C_t \xi_c B_c - C_t \bar{D} \xi_c B_c]. \quad (8)$$

With this form (6) of the BRST operator, it is obvious that the cohomology of the Q_{sN4} is a direct product of those of $Q_{N=2}$ and Q_{top} . The BRST operator Q_{top} imposes the condition that the fields $(C_c, B_c, \xi_c, \beta_c)$ as well as $(C_a, B_a, \xi_a, \beta_a)$ decouple from the physical sector and the cohomology of the operator consists only of their vacuum. Thus we obtain one-to-one correspondence of the cohomologies of Q_{sN4} and $Q_{N=2}$, and the small $N = 4$ string propagating in the background described by (3) is equivalent to the $N = 2$ string. It has also been shown that this phenomenon can be interpreted as a spontaneous breakdown of superconformal symmetry [5].

Let us next turn to the embedding of the small $N = 4$ into the large $N = 4$ superstring. In order to discuss this embedding, it turns out convenient to express the $N = 4$ SCA in $N = 1$ superfields [6]:

$$\begin{aligned}
T(Z_1)\mathcal{O}(Z_2) &\sim \frac{\frac{c}{6}}{z_{12}^3}T + \frac{3}{2}\frac{\theta_{12}}{z_{12}^2}T + \frac{1}{2}\frac{1}{z_{12}}DT + \frac{\theta_{12}}{z_{12}}\partial T, \\
T(Z_1)\mathcal{O}(Z_2) &\sim h_{\mathcal{O}}\frac{\theta_{12}}{z_{12}^2}\mathcal{O} + \frac{1}{2}\frac{1}{z_{12}}D\mathcal{O} + \frac{\theta_{12}}{z_{12}}\partial\mathcal{O}, \\
T(Z_1)H(Z_2) &\sim \frac{\frac{c}{12} + \frac{1}{2}\theta_{12}H}{z_{12}^2} + \frac{1}{2}\frac{1}{z_{12}}DH + \frac{\theta_{12}}{z_{12}}\partial H, \\
C_3(Z_1)C_{\pm}(Z_2) &\sim \pm \frac{2C_{\pm} + \theta_{12}DC_{\pm}}{z_{12}}, \quad C_3(Z_1)C_3(Z_2) \sim \frac{\frac{c}{3}}{z_{12}^2} + \frac{\theta_{12}2T}{z_{12}}, \\
C_+(Z_1)C_-(Z_2) &\sim \frac{\frac{2}{3}c}{z_{12}^2} + \frac{4C_3 + \theta_{12}(4T + 2DC_3)}{z_{12}}, \\
N_3(Z_1)C_{\pm}(Z_2) &\sim N_{\pm}(Z_1)C_3(Z_2) \sim \mp \frac{N_{\pm} + \theta_{12}C_{\pm}}{z_{12}}, \\
N_3(Z_1)C_3(Z_2) &\sim -\frac{\frac{c}{6}\theta_{12}}{z_{12}^2} - \frac{H}{z_{12}}, \quad N_{\pm}(Z_1)C_{\mp}(Z_2) \sim \frac{\frac{c}{3}\theta_{12}}{z_{12}^2} + \frac{2(H \mp N_3) \pm \theta_{12}2C_3}{z_{12}}, \\
N_3(Z_1)N_{\pm}(Z_2) &\sim \frac{\mp\theta_{12}}{z_{12}}N_{\pm}, \quad N_+(Z_1)N_-(Z_2) \sim \frac{k}{z_{12}} - \frac{\theta_{12}}{z_{12}}2N_3, \\
N_3(Z_1)N_3(Z_2) &\sim \frac{\frac{k}{2}}{z_{12}}, \quad H(Z_1)C_{\pm}(Z_2) \sim \frac{N_{\pm}}{z_{12}}, \\
H(Z_1)C_3(Z_2) &\sim -\frac{N_3}{z_{12}}, \quad H(Z_1)H(Z_2) \sim -\frac{\frac{k}{2}}{z_{12}}, \tag{9}
\end{aligned}$$

where $Z \equiv (z, \theta)$, $D \equiv \partial_{\theta} + \theta\partial_z$ is the $N = 1$ covariant derivative, $z_{12} \equiv z_1 - z_2 + \theta_1\theta_2$, $\theta_{12} \equiv \theta_1 - \theta_2$ and the operators \mathcal{O} stand for $C_{\pm}, C_3, N_{\pm}, N_3$ with their dimensions

$h_C = 1, h_N = \frac{1}{2}$, respectively.¹ Here T, N, H are fermionic superfields, C are bosonic and the central charge c and k are given as

$$c = 3k_1, \quad k = k_1 + k_2, \quad (10)$$

in terms of the levels k_1, k_2 of the two $SU(2)$ current algebras in the large $N = 4$ SCA. Notice that the generators T, C make a closed algebra which is the small $N = 4$ SCA. It is interesting to notice that the algebra can be expressed without explicit dependence on the ratio of the levels of current algebras.

We note that here is a very special situation different from all embeddings known so far [1, 2]. Since the small $N = 4$ SCA is a subalgebra of the large $N = 4$, the OPEs of unbroken generators with unbroken ones close as usual.² In the usual embeddings, the OPEs of unbroken generators with broken ones give broken ones. However, in our special case, the OPE of C with N involves C . This makes our embedding very special.

In the usual case, all the unbroken generators are again themselves unbroken generators in the theory with higher symmetries. However, here this cannot be the case because they are independent of the additional matter fields and cannot have nonzero OPE with N generators which consist of solely new additional matter fields. It turns out that the generators C need an infinite number of terms dependent on the new fields, just as the non-critical embeddings found in ref. [8]. Remarkably we can determine the explicit structure of these infinite terms from the consistency of the OPE (9).

Now our realization is as follows. We take generators T_m, C_m which satisfy the small

¹ The component structure of the $N = 1$ superfields in the notation of ref. [7] is as follows:

$$\begin{aligned} T &= \frac{1}{2}(G_{(1,2)} + G_{(2,1)}) + \theta T, \quad C_+ = 2H_+ + 2\theta[G_{(1,1)} + (1+x)F'_{(1,1)}], \\ C_- &= 2H_- - 2\theta[G_{(2,2)} + (1+x)F'_{(2,2)}], \quad C_3 = 2H_3 + \theta[G_{(2,1)} - G_{(1,2)} - (1+x)F'_{(1,2)} + (1+x)F'_{(2,1)}], \\ N_+ &= 2F_{(1,1)} + \theta(H_+ - J_+), \quad N_- = -2F_{(2,2)} + \theta(H_- - J_-), \\ N_3 &= F_{(1,2)} - F_{(2,1)} - \theta(H_3 + J_3), \quad H = F_{(1,2)} + F_{(2,1)} + \theta J, \end{aligned}$$

where G, H, J and F are the super generators, two $SU(2)$ currents and fermionic currents, respectively.

²We are referring as “unbroken” to those generators that remain exact in the broken phase ($N = 2$ superconformal generator T in our previous example) and to other generators as “broken”.

$N = 4$ SCA in (9) with $c = -12$ and introduce additional matter fields $(b^+, \gamma_+), (b^-, \gamma_-), (b^3, \gamma_3), (b^0, \gamma_0)$ of spins $(\frac{1}{2}, 0)$. They have the OPE

$$\gamma_i(Z_1)b^j(Z_2) \sim \delta_i^j \frac{\theta_{12}}{z_{12}}. \quad (11)$$

Here and in what follows, we denote commuting fields by β, γ and anticommuting ones by b, c unless otherwise stated. We then find that the following generators satisfy the large $N = 4$ algebra (9) with $c = 0, k = 0$:

$$\begin{aligned} T &= T_m + \frac{1}{2}b^+\partial\gamma_+ + \frac{1}{2}Db^+D\gamma_+ + \frac{1}{2}b^-\partial\gamma_- + \frac{1}{2}Db^-D\gamma_- \\ &\quad + \frac{1}{2}b^3\partial\gamma_3 + \frac{1}{2}Db^3D\gamma_3 + \frac{1}{2}b^0\partial\gamma_0 + \frac{1}{2}Db^0D\gamma_0 + \sum_{n=1}^{\infty} T_n, \\ C_+ &= C_{m,+} + 2b^-D\gamma_3 + 2b^0D\gamma_+ + \sum_{n=1}^{\infty} C_{+,n}, \\ C_- &= C_{m,-} + 2b^3D\gamma_- + 2b^+D\gamma_0 + \sum_{n=1}^{\infty} C_{-,n}, \\ C_3 &= C_{m,3} - b^+D\gamma_+ + b^-D\gamma_- - b^3D\gamma_3 + b^0D\gamma_0 + \sum_{n=1}^{\infty} C_{3,n}, \\ N_+ &= b^- - b^-\gamma_3 - b^0\gamma_+, \\ N_- &= b^+ - b^+\gamma_0 - b^3\gamma_-, \\ N_3 &= \frac{1}{2}(b^3 - b^0 - b^+\gamma_+ + b^-\gamma_- - b^3\gamma_3 + b^0\gamma_0), \\ H &= \frac{1}{2}(-b^3 - b^0 + b^+\gamma_+ + b^-\gamma_- + b^3\gamma_3 + b^0\gamma_0), \end{aligned} \quad (12)$$

where

$$\begin{aligned} T_n &= -D\partial \left[\frac{1}{n}\gamma_3^n + \frac{1}{n}\gamma_0^n + \sum_{l,k} \frac{l}{(l!)^2} \frac{(k+l-1)!(n-k-l-1)!}{k!(n-k-2l)!} (\gamma_+\gamma_-)^l \gamma_3^{n-2l-k} \gamma_0^k \right], \\ C_{+,n} &= \left[\gamma_0^n - \gamma_0^{n-1}\gamma_3 + (n-1)\gamma_+\gamma_-\gamma_0^{n-2} \right. \\ &\quad \left. + \sum_{l,k} \frac{l(l-1)}{(l!)^2} \frac{(k+l-2)!(n-k-l)!}{k!(n-k-2l)!} (\gamma_+\gamma_-)^l \gamma_0^{n-2l-k} \gamma_3^k \right] C_{m,+} \\ &\quad - \sum_{l,k} \frac{1}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} \gamma_+^{l+2} \gamma_-^l \gamma_3^{n-2-2l-k} \gamma_0^k C_{m,-} \\ &\quad - 2\gamma_+ \left[\gamma_0^{n-1} + \sum_{l,k} \frac{l}{(l!)^2} \frac{(k+l-1)!(n-1-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l \gamma_0^{n-1-2l-k} \gamma_3^k \right] C_{m,3} \end{aligned}$$

$$\begin{aligned}
& -4(\partial\gamma_+)\gamma_0^{n-1} - \sum_{l,k} \frac{4}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} \gamma_+^{l+1} \gamma_-^l (\partial\gamma_3) \gamma_3^{n-2-2l-k} \gamma_0^k \\
& - \sum_{l,k} \frac{4l}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l (\partial\gamma_+) \gamma_3^{n-1-2l-k} \gamma_0^k, \\
C_{-,n} = & - \sum_{l,k} \frac{1}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} \gamma_+^l \gamma_-^{l+2} \gamma_0^{n-2-2l-k} \gamma_3^k C_{m,+} \\
& + \left[\gamma_3^n - \gamma_3^{n-1} \gamma_0 + (n-1) \gamma_+ \gamma_- \gamma_3^{n-2} \right. \\
& \left. + \sum_{l,k} \frac{l(l-1)}{(l!)^2} \frac{(k+l-2)!(n-k-l)!}{k!(n-k-2l)!} (\gamma_+\gamma_-)^l \gamma_3^{n-2l-k} \gamma_0^k \right] C_{m,-} \\
& + 2\gamma_- \left[\gamma_3^{n-1} + \sum_{l,k} \frac{l}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l \gamma_0^{n-1-2l-k} \gamma_3^k \right] C_{m,3} \\
& - 4(\partial\gamma_-) \gamma_3^{n-1} - \sum_{l,k} \frac{4}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} \gamma_+^l \gamma_-^{l+1} (\partial\gamma_0) \gamma_0^{n-2-2l-k} \gamma_3^k \\
& - \sum_{l,k} \frac{4l}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l (\partial\gamma_-) \gamma_0^{n-1-2l-k} \gamma_3^k, \\
C_{3,n} = & -\gamma_- \left[\gamma_0^{n-1} + \sum_{l,k} \frac{l}{(l!)^2} \frac{(k+l-1)!(n-1-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l \gamma_0^{n-1-2l-k} \gamma_3^k \right] C_{m,+}, \\
& + \gamma_+ \left[\gamma_3^{n-1} + \sum_{l,k} \frac{l}{(l!)^2} \frac{(k+l-1)!(n-1-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l \gamma_3^{n-1-2l-k} \gamma_0^k \right] C_{m,-}, \\
& + \sum_{l,k} \frac{2}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} (\gamma_+\gamma_-)^{l+1} \gamma_0^{n-2-2l-k} \gamma_3^k C_{m,3} \\
& - 2(\partial\gamma_0) \gamma_0^{n-1} + 2(\partial\gamma_3) \gamma_3^{n-1} + \sum_{l,k} \frac{2l}{(l!)^2} \frac{(k+l-1)!(n-1-k-l)!}{k!(n-1-k-2l)!} (\gamma_+\gamma_-)^l \\
& \quad \times \left[(\partial\gamma_3) \gamma_3^{n-1-2l-k} \gamma_0^k - (\partial\gamma_0) \gamma_0^{n-1-2l-k} \gamma_3^k \right] \\
& + \sum_{l,k} \frac{2}{(l!)^2} \frac{(k+l)!(n-2-k-l)!}{k!(n-2-k-2l)!} (\gamma_+\gamma_-)^l \left[(\partial\gamma_+) \gamma_- \gamma_3^{n-2-2l-k} \gamma_0^k \right. \\
& \quad \left. - \gamma_+ (\partial\gamma_-) \gamma_0^{n-2-2l-k} \gamma_3^k \right], \tag{13}
\end{aligned}$$

where the sums over k and l run over non-negative integers.

The large $N = 4$ BRST operator is given by

$$\begin{aligned}
Q_{N4} = & \int \frac{dzd\theta}{2\pi i} [c_t T + \gamma_{c+} C_+ + \gamma_{c-} C_- + \gamma_{c3} C_3 + c_{n+} N_+ + c_{n-} N_- + c_{n3} N_3 + c_h H \\
& + c_t \left[\frac{1}{2} T_{3/2}(\beta^t, c_t) + T_1(b^{c+}, \gamma_{c+}) + T_1(b^{c-}, \gamma_{c-}) + T_1(b^{c3}, \gamma_{c3}) + T_{1/2}(\beta^{n+}, c_{n+}) \right]
\end{aligned}$$

$$\begin{aligned}
& +T_{1/2}(\beta^{n-}, c_{n-}) + T_{1/2}(\beta^{n3}, c_{n3}) + T_{1/2}(\beta^h, c_h) \Big] \pm 2(D\gamma_{c3})b^{c\pm}\gamma_{c\pm} \pm \gamma_{c3}(Db^{c\pm})\gamma_{c\pm} \\
& -\beta^t(\gamma_{c3})^2 + 4(D\gamma_{c+})b_{c3}\gamma_{c-} - 4\gamma_{c+}\beta^t\gamma_{c-} + 2\gamma_{c+}(Db^{c3})\gamma_{c-} + (Dc_{n3})\beta^h\gamma_{c3} \\
& \pm(Dc_{n3})\beta^{n\pm}\gamma_{c\pm} \pm c_{n3}b^{c\pm}\gamma_{c\pm} \pm (Dc_{n\pm})\beta^{n\pm}\gamma_{c3} \pm c_{n\pm}b^{c\pm}\gamma_{c3} - 2(Dc_{n\pm})(\beta^h \mp \beta^{n3})\gamma_{c\mp} \\
& \mp 2c_{n\pm}b^{c3}\gamma_{c\mp} \mp c_{n3}\beta^{n\pm}c_{n\pm} - 2c_{n+}\beta^{n3}c_{n-} - Dc_h\beta^{n\pm}\gamma_{c\pm} + Dc_h\beta^{n3}\gamma_{c3} \Big], \quad (14)
\end{aligned}$$

where the ghosts for each generators are labelled by their corresponding subscripts and superscripts, and

$$T_j(b, c) = (-1)^{\epsilon_b+1}jb\partial c + (-1)^{\epsilon_b}\left(\frac{1}{2} - j\right)\partial bc + \frac{1}{2}DbDc, \quad (15)$$

is the energy-momentum tensor for the (b, c) -system of spin $(j, \frac{1}{2} - j)$. In eq. (15), the statistics of the fields (b, c) are not restricted except that b and c have opposite characters, and ϵ_b is odd (even) for anticommuting (commuting) b . Substituting the generators (12) into (14), we obtain the BRST operator for this system which contains an infinite number of terms.

We now show that the BRST operator can be mapped by a series of similarity transformations into the sum of those for the small $N = 4$ string Q_{sN4} and topological sector Q_{top} :

$$Q_{sN4} + Q_{top}, \quad (16)$$

where

$$\begin{aligned}
Q_{sN4} &= \int \frac{dzd\theta}{2\pi i} [c_t T + \gamma_{c+}C_+ + \gamma_{c-}C_- + \gamma_{c3}C_3 \\
&+ c_t \left(\frac{1}{2}T_{3/2}(\beta_t, c_t) + T_1(b^{c+}, \gamma_{c+}) + T_1(b^{c-}, \gamma_{c-}) + T_1(b^{c3}, \gamma_{c3}) \right) \\
&\pm 2(D\gamma_{c3})b^{c\pm}\gamma_{c\pm} \pm \gamma_{c3}Db^{c\pm}\gamma_{c\pm} - \beta^t(\gamma_{c3})^2 + 4(D\gamma_{c+})b^{c3}\gamma_{c-} \\
&- 4\gamma_{c+}\beta^t\gamma_{c-} + 2\gamma_{c+}(Db^{c3})\gamma_{c-} \Big], \\
Q_{top} &= \int \frac{dzd\theta}{2\pi i} \left[c_{n+}b^- + c_{n-}b^+ + \frac{1}{2}c_{n3}(b^3 - b^0) - \frac{1}{2}c_h(b^3 + b^0) \right]. \quad (17)
\end{aligned}$$

To show this, we first transform it by

$$R_1 = \int \frac{dzd\theta}{2\pi i} [\{\gamma_{c+}(\gamma_3 - \gamma_0) + \gamma_{c3}\gamma_-\}b^{c+} - \{\gamma_{c-}(\gamma_3 - \gamma_0) + \gamma_{c3}\gamma_+\}b^{c-}$$

$$\begin{aligned}
& +2(\gamma_{c+}\gamma_+ - \gamma_{c-}\gamma_-)b^{c3} - (\gamma_3 + \gamma_0)(\gamma_+b^+ + \gamma_-b^-) - (\gamma_0^2 + \gamma_+\gamma_-)b^0 - (\gamma_3^2 + \gamma_+\gamma_-)b^3 \\
& + \left\{ \gamma_0 c_{n+} + \frac{1}{2}\gamma_-(c_{n3} - c_h) \right\} \beta^{n+} + \left\{ \gamma_3 c_{n-} - \frac{1}{2}\gamma_+(c_{n3} + c_h) \right\} \beta^{n-} \\
& + \left\{ \gamma_+ c_{n+} - \gamma_- c_{n-} + \frac{1}{2}\gamma_3(c_{n3} - c_h) + \frac{1}{2}\gamma_0(c_{n3} + c_h) \right\} \beta^{n3} \\
& - \left\{ \gamma_+ c_{n+} + \gamma_- c_{n-} + \frac{1}{2}\gamma_3(c_{n3} - c_h) - \frac{1}{2}\gamma_0(c_{n3} + c_h) \right\} \beta^h \Big]. \tag{18}
\end{aligned}$$

Remarkably this transformation brings the total BRST operator into a sum of finite number of terms. The next transformation generated by

$$\begin{aligned}
R_2 = \int \frac{dzd\theta}{2\pi i} & \left[\{ \gamma_{c3}(\beta^{n3} + \beta^h) - 2\gamma_{c-}\beta^{n-} \} D\gamma_0 + \{ \gamma_{c3}(\beta^{n3} - \beta^h) - 2\gamma_{c+}\beta^{n+} \} D\gamma_3 \right. \\
& \left. + \{ \gamma_{c3}\beta^{n-} + 2\gamma_{c+}(\beta^{n3} + \beta^h) \} D\gamma_+ - \{ \gamma_{c3}\beta^{n+} + 2\gamma_{c-}(\beta^{n3} - \beta^h) \} D\gamma_- \right], \tag{19}
\end{aligned}$$

then eliminates irrelevant cubic terms from the BRST operator. A final transformation by

$$\begin{aligned}
R_3 = \int \frac{dzd\theta}{2\pi i} & \frac{1}{2} c_t [\beta^{n+} \partial \gamma_- - D\beta^{n+} D\gamma_- + \beta^{n-} \partial \gamma_+ - D\beta^{n-} D\gamma_+ + \beta^{n3} \partial (\gamma_3 - \gamma_0) \\
& - D\beta^{n3} D(\gamma_3 - \gamma_0) - \beta^h \partial (\gamma_3 + \gamma_0) + D\beta^h D(\gamma_3 + \gamma_0)], \tag{20}
\end{aligned}$$

maps it to the sum of $Q_{sN4} + Q_{top}$. In the same manner as the $N = 2$ string, the BRST operator Q_{top} imposes the condition that the fields $(c_{n\pm}, \beta^{n\pm}, \gamma_{\mp}, b^{\mp}), (c_{n3}, \beta^{n3}, \gamma_3 - \gamma_0, \frac{1}{2}(b^3 - b^0)), (c_h, \beta^h, \gamma_3 + \gamma_0, \frac{1}{2}(b^3 + b^0))$ all fall into the quartet representation of the BRST operator and hence decouple from the physical sector. Thus the large $N = 4$ superstring in our particular background is equivalent to the small $N = 4$ superstring.

Finally let us discuss if the general large $N = 4$ superstrings can be embedded into higher superstrings in the spirit of the universal string. It has been shown in ref. [7] that the critical condition for the large $N = 4$ superstring is that the total central charge is zero but that the ratio x of the levels of the two $SU(2)$ current algebras in its SCA can be arbitrary. In ref. [2], those with the equal levels were embedded into the hierarchy of superstrings. The question thus remains if the general large $N = 4$ superstring with $c = 0$ but an arbitrary x can be embedded into the $N = 5$ superstrings. However, as can be seen from eq. (9), the critical $N = 4$ SCA can be written in the same form for an arbitrary ratio of the levels just by changing the correspondence with the component generators.

We have checked that this is true also when it is written in $N = 2$ superfields. Hence after this appropriate redefinitions of the superfield generators, the realization given in ref. [2] is actually valid for an arbitrary ratio of the levels.

This completes the program of embedding all the superstrings into the unified framework of the universal string.

Acknowledgements

We have checked many of our super-OPEs using the Mathematica package developed by S. Krivonos and K. Thielemans, whose software is gratefully acknowledged. We would also like to thank H. Kunitomo for valuable discussions. In addition, we thank J. O. Madsen and J. de Boer for useful advice on the use of OPE package.

References

- [1] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A9** (1994) 653.
- [2] F. Bastianelli, N. Ohta and J. L. Petersen, Phys. Lett. **B327** (1994) 35; Phys. Rev. Lett. **73** (1994) 1199.
- [3] M. Ademollo et al., Phys. Lett. **62B** (1976) 105; Nucl. Phys. **B114** (1976) 297.
- [4] H. Ishikawa and M. Kato, Mod. Phys. Lett. **A9** (1994) 725;
F. Bastianelli, Phys. Lett. **B322** (1994) 340;
N. Ohta and J. L. Petersen, Phys. Lett. **B325** (1994) 67.
- [5] H. Kunitomo, Phys. Lett. **B343** (1995) 144.
- [6] A. Sevrin and G. Theodoridis, Nucl. Phys. **B332** (1990) 380.
- [7] F. Bastianelli and N. Ohta, Phys. Rev. **D50** (1994) 4051.
- [8] N. Berkovits and N. Ohta, Phys. Lett. **B334** (1994) 72.